

# Discontinuous piecewise polynomial collocation in two dimensions

John Loustau\*, Scott Irwin, Ariel Lindorff and John Svadlenka  
*Hunter College, New York, NY, USA*

**Abstract.** We consider collocation method based on a piecewise polynomial approach in two spatial dimensions. Our model uses a finite element partition where the interpolation may be discontinuous across element boundaries. The underlying topological structure is the continuous or bounded functions with sup-norm. To ensure broad applicability we use a triangular finite element partition. We include a convergence result and examples. The examples include comparisons to finite element method and examples from finance.

Keywords: Collocation method, finite element method, piecewise polynomial interpolation, triangulation

## 1. Introduction

In [6] the author states that collocation method (CM) is the technique of choice in finance and economics. Other authors state that the method is only useful for equations of one spatial dimension; that is, for a PDE with a single variable and time. It is our intention to resolve these competing points of view for the two-dimensional case.

On the one hand, CM provides a more immediate, intuitively apparent and flexible numerical technique for PDE. As viewed against finite difference method (FDM), CM and finite element method (FEM) support unrestricted partitions of the spatial domain. As viewed against FEM, CM uses the PDE, not the weak form of the equation. When implemented, CM does not require that we compute integrals. CM, as implemented here, does use a partial assembly process. As in the case of FEM, the purpose of the assembly is to achieve a square linear system. As with FEM, the model function is continuous at points associated to the assembly. In addition, this implementation is significantly less memory intensive than FEM. Therefore, we are able to implement finer partitions on smaller computers. In summary, CM is easier to understand, easier to program and is more efficient.

As we see below, CM based on piecewise polynomial interpolation is supported by a convergence theory. As in [1], convergence occurs in the normed space

of bounded functions on a compact set. Hence, the underlying theory is distinct from the standard approach for FEM. In order to support a wide variety of regions, we focus on triangular finite element partitions. In particular, this treatment of CM is distinct from the treatment presented in [2]. We consider two models based on six nodal triangles, one with 21 degrees of freedom (DoF) and one with 6 DoF. In either case, the model is distinct from the one used in FEM. The implementation yields a space of discontinuous functions on the domain of the PDE. Indeed, the title refers to this. As in [4], the resulting linear system is over specified. Unlike [4] we accumulate at the vertex nodes thereby achieving a square linear system.

We organize our presentation as follows. First we develop the necessary terminology and notation. In this work we focus on the two dimensional case with a triangular element partition. In Section 3, we present the piecewise polynomial interpolation operator. The supporting theory is developed in Section 3. In Section 4 we define collocation method and present the solution procedures. The convergence theory is presented in Section 5. These proofs are largely unchanged from the corresponding results in [1]. We conclude this section with a discussion of the relationship between the CM solution and the FEM solution. In Section 6 we present examples to compare CM results to corresponding FEM results. Finally, in Section 7 we consider familiar examples from mathematical finance. We look specifically at options pricing for barrier options. In particular we consider the Black–Scholes equation on a triangular and trapezoidal region. These regions are necessary

---

\* Address for correspondence: John Loustau, Hunter College, New York, NY 10065, USA. E-mail: jloustau@msn.com.

for barrier options and may only be implemented with triangular elements. In addition we look at the mean-exit-time equation from a price corridor as an estimator of the practical life of an option.

## 2. Terminology and notation

Throughout  $D$  denotes a compact subset of  $\mathbb{R}^2$ . We will only consider real valued functions on  $D$ . Further,  $D$  has a finite element partition. We denote the elements by  $E_e$ , the set of vertices by  $P$ , the interior of the elements by  $U_e$  and the union of the  $U_e$  as  $U$ . To insure that the partition reasonably approximates the actual boundary of  $D$ , we will suppose that the elements are triangles. For simplicity we suppose that  $D = \bigcup_e E_e$ . The parameter  $h$  is the maximal side length of any triangle in the partition and referred to as the *mesh parameter*.

We consider the triangle with vertices  $(0, 0)$ ,  $(1, 0)$  and  $(0, 1)$  as our reference element and denote this triangle as  $T$ . We select a finite set of points of  $T$  and denote this set together with the three vertices as  $Y$ . We are concerned with two cases,  $|Y| = 6$  or  $21$ . From [5],  $Y$  determines a basis of polynomials for the space of polynomials up to degree 2 when  $Y$  has 6 elements, and degree 5 when  $Y$  has 21 elements. We denote this space by  $\Pi_n$ ,  $n = 2$  or  $5$ . Given a finite set of functions  $\{f_i\}$  and points  $\{x_i\}$ , we say that the functions and points are *dual* provided  $f_i(x_j) = \delta_{i,j}$ . In particular, we select a basis  $\{\phi_j\}$  of  $\Pi_n$  that is dual to  $Y$  in this sense.

For each  $E_e$  there is an affine mapping  $A_e: T \rightarrow E_e$ . The affine mapping identifies polynomials  $\phi_{e,j} = \phi_j A_e$  on each element. These polynomials are dual to  $A_e(Y) = Z_e$ . We denote the corresponding polynomial space by  $V_e^h$ . In the usual terminology for CM,  $Z = \bigcup Z_e$  is the set of *collocation points*. Note that, as in the one or two-dimensional cases for orthogonal spline collocation, there are only  $|Y|$  collocation points

that are distributed by the affine maps to each  $E_e$  in  $D$ . This fact is essential to the theory.

Note that the basis polynomials may be joined across element boundaries to define piecewise polynomial functions that are continuous on  $U \cup P$ . More precisely, if  $E_e$  and  $E_{e'}$  are adjacent, then for each  $j$  we may join  $\phi_{e,j}$  and  $\phi_{e',j}$  to form a continuous function on  $U \cup P$ . We denote the space of these functions by  $V^h$ . It should be noted that we will not force an element of  $V^h$  to be continuous along the common boundary of two elements even if a collocation point, not in  $P$ , lies on the common boundary.

Alternately,  $V_e^h$  has a basis  $\{N_{i,\alpha_i}^e\}$  of polynomials that is dual to the usual DoF of the FEM models for the 6 nodal triangle (see Fig. 1a). This alternate basis for the degree 5 case may be used to determine boundary forces for CM.

A triangular finite element partition may be refined by connecting the midpoints of each element as in Fig. 1a and b. The process divides each triangle into 4 congruent triangles where each triangle is similar to the original by a ratio of 1–2. Therefore, the mesh parameter through successive refinements process will go to zero. Later when we refer to convergence as  $h \rightarrow 0$ , we are referring to this refinement process. In the terminology of FEM, this refinement process is *quasi-regular*.

## 3. Piecewise polynomial interpolation

In this section we develop the basic theory of piecewise polynomial interpolation within the context developed in Section 2. The viewpoint extends the treatment presented in [1,4].

We begin with the space  $V$  of bounded functions on  $D$  which are  $C^0(U \cup P)$  and extendable to a continuous function of  $E_e$  for each element. The latter condition is satisfied provided the function  $v$  has limits  $\text{Lim}_{x \rightarrow a} v(x)$  for  $a$  in  $E_e$  and  $x$  in the interior of  $E_e$  for each element. The norm will be *sup norm* denoted

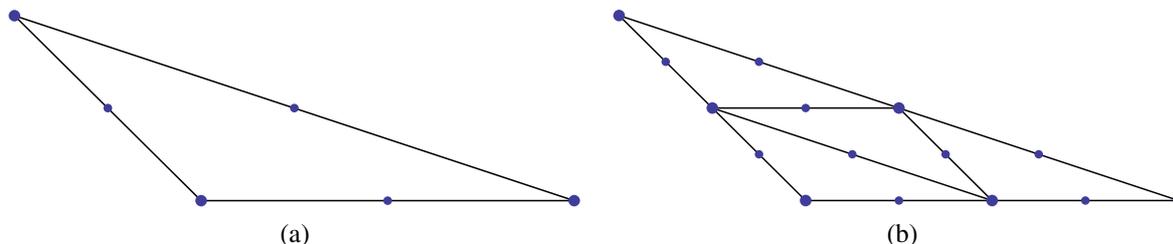


Fig. 1. (a) Six nodal triangle. (b) Triangle partitioned into 4 congruent triangles. (Colors are visible in the online version of the article; <http://dx.doi.org/10.3233/RDA-2012-0076>.)

$\|v\|_\infty$ . Note that this space contains  $V^h$  as a finite dimensional subspace. In addition  $V$  is a Banach space. Indeed, a Cauchy sequence in  $V$  induces Cauchy sequences in each  $E_e$ .

For each element  $E_e$  we define the usual polynomial interpolation operator  $\mathcal{P}_{Z_e}^h : C^0(E_e) \rightarrow V_e^h$  with respect to the basis polynomials  $\phi_j$ . Since the triangle vertices are included in the sets  $Z_e$ , then these operators may be joined to form an linear projection  $\mathcal{P}_Z^h : V \rightarrow V^h$ , where  $Z$  is the union of the  $Z_e$ . We refer to  $\mathcal{P}_Z^h$  as the *piecewise polynomial interpolation operator*. We prove that for fixed  $Y$ , this operator is bounded independent of the finite element partition.

**Theorem 3.1.** *The operator  $\mathcal{P}_Z^h$  is bounded with  $\|\mathcal{P}_Z^h\| \leq M$ , dependent on  $Y$  and independent of  $h$ .*

**Proof.** Take  $u$  in  $V$  with  $\|u\|_\infty \leq 1$ . We claim that  $\|\mathcal{P}_Z^h u\|_\infty \leq M$  for  $M$  independent of the finite element partition. If we denote the sup norm of  $\mathcal{P}_Z^h u$  restricted to  $E_e$  by  $\|\mathcal{P}_{Z_e}^h u\|_{e,\infty}$ , then it suffices to prove that  $\|\mathcal{P}_{Z_e}^h u\|_{e,\infty} \leq M$  for each  $e$ .

Using the notation developed above,  $\mathcal{P}_{Z_e}^h u$  restricted to  $E_e$  is  $\mathcal{P}_{Z_e}^h u$  and may be written as  $\sum_j a_j \phi_{e,j}$ . Taking the norm, we have  $\|\mathcal{P}_{Z_e}^h u\|_{e,\infty} = \|\mathcal{P}_{Z_e}^h u\|_{e,\infty} \leq \sum_j |a_j| \|\phi_{e,j}\|_{e,\infty}$ .

Since the  $\phi_{e,j}$  are dual to  $Z_e$ , then  $a_j = u(z_{e,j})$  where  $z_{e,j} = A_e(y_j)$ . Hence,  $|a_j| \leq 1$  for each  $j$  and we have  $\|\mathcal{P}_{Z_e}^h u\|_{e,\infty} \leq \sum_j \|\phi_{e,j}\|_{e,\infty}$ . Since the values of  $\phi_{e,j}$  on  $E_e$  are the same as the corresponding values of  $\phi_j$  on  $T$ , then  $\|\phi_{e,j}\|_{e,\infty}$  is equal to the maximum of  $|\phi_j(x)|$  for  $x$  in  $T$ . If we set  $\gamma_j = \max_{x \in T} |\phi_j(x)|$ , then we have  $\|\mathcal{P}_{Z_e}^h u\|_{e,\infty} \leq \sum_j \gamma_j$  independent of the finite element partition. Hence,  $M$  exists with  $M \leq \sum_j \gamma_j$ .  $\square$

We now prove that the piecewise polynomial interpolation of a twice continuously differentiable function  $u$  converges to  $u$  as  $h \rightarrow 0$ . As stated, the convergence is uniform of order  $h^2$ .

**Theorem 3.2.** *If  $u$  in  $V$  is  $C^2$  on each element, then  $\mathcal{P}_Z^h u$  converges uniformly to  $u$  as  $h^2$  goes to zero.*

**Proof.** As above, it suffices to prove the result on each  $E_e$ . Let  $p$  denote the degree 1 interpolation of  $u$  on the element. Since  $p \in V_e^h$  then  $\mathcal{P}_{Z_e}^h(p) = p$ . Restricting  $u$

to the element,

$$\begin{aligned} & \|u - \mathcal{P}_Z^h u\|_{e,\infty} \\ &= \|u - \mathcal{P}_{Z_e}^h u\|_{e,\infty} \\ &\leq \|u - p\|_{e,\infty} + \|p - \mathcal{P}_{Z_e}^h u\|_{e,\infty} \\ &= \|u - p\|_{e,\infty} + \|\mathcal{P}_{Z_e}^h p - \mathcal{P}_{Z_e}^h u\|_{e,\infty} \\ &\leq (1 + \|\mathcal{P}_{Z_e}^h\|) \|u - p\|_{e,\infty}. \end{aligned}$$

Since  $u$  is  $C^2$ , then by [5] there is a constant  $K$  depending only on  $D$  and  $u$  so that

$$\|u - p\|_{e,\infty} \leq Kh^2.$$

By Theorem 3.1,  $\|\mathcal{P}_Z^h\| \leq M$  is independent of the particular finite element partition. Therefore, it follows that  $\|u - \mathcal{P}_Z^h u\|_\infty \leq (1 + M)Kh^2$ .  $\square$

#### 4. CM via piecewise polynomial interpolation

For an order 2, linear PDE we will consider expressions  $Lv + f$ , where  $L$  is linear and  $L : C^2(E_e) \rightarrow C^0(E_e)$  for each element and  $f$  is in  $C^0(E_e)$ . We will suppose that the PDE has a solution  $u$  in  $V$  that is twice continuously differentiable on each element. We now define collocation.

**Definition 4.1.** For the given model, the *discontinuous piecewise polynomial interpolation CM solution* to  $Lv + f$  is an element  $u_P$  of  $V^h$  satisfying for each element

$$\mathcal{P}_Z^h L u_P - \mathcal{P}_Z^h f = 0. \quad (4.1)$$

Note that (4.1) is equivalent to

$$(L u_P - f)(z_{e,j}) = 0 \quad \text{for every } e \text{ and } j. \quad (4.2)$$

The last expression is the system of linear equations that we associate to collocation method. If  $u_P = \sum_{e,k} a_{e,k} \phi_{e,k}$ , then at the level of an element, the linear system with unknowns  $a_{e,k}$  is

$$\sum_k a_{e,k} (L \phi_{e,k})(z_{e,i}) = f(z_{e,i}) \quad (4.3)$$

for each pair  $e, i$ .

However, this linear system as stated is over specified as it does not take into account the continuity at the element vertices. Specifically, if  $m$  is the number of elements and  $n$  is the number of DoF, then there are  $n$  times  $m$  equations in (4.2). But if  $E_e$  and  $E_{e'}$  share a vertex and if  $\phi_{e,k}$  and  $\phi_{e',k'}$  are dual to the common vertex, then  $a_{e,k} = a_{e',k'}$ . Hence, there are only  $(n - 3) \times m + p$  variables to be resolved.

In [4] the authors do consider the over specified alternative via least squares approximation. By restricting their attention to the Poisson equation and supposing that the bilinear form derived from the least squares fitting satisfies the elliptical hypothesis, then they are able to prove convergence.

We present an alternate procedure to resolve the system (4.2). In particular we use an assembly procedure familiar from FEM. In our case the over specification occurs when we assemble the systems in (4.3) into a single system as the sets of collocation points are not disjoint. If the rows are labeled  $e, i$  and columns labeled  $e', k$ , then at a common vertex we have  $z_{e,i} = z_{e',i'}$ . Hence, the associated linear relations

$$\sum_k a_{e,k}(L\psi_{e,k})(z_{e,i}) = f(z_{e,i})$$

and

$$\sum_k a_{e',k'}(L\psi_{e',k'})(z_{e,i}) = f(z_{e,i})$$

may be accumulated to a single row. If we repeat this procedure at each vertex collocation point, then the result is a square  $[(n - 3)m + p] \times [(n - 3) \times m + p]$  linear system.

Rendering this procedure into steps we have:

- A1 Select the point set  $Y$  in  $T$ , determine the  $\phi_k$  on  $T$  and the affine maps  $A_e$ .
- A2 Use the  $A_e$  to determine the polynomials  $\psi_{e,k}$  and the collocation point sets  $Z_e$ . For storage considerations, hold these as functions of the element vertices to be specified as necessary.
- A3 Set up the individual linear relations to write  $u_P$  on the element in terms of the residual of the  $\phi_{e,k}$  (see Eq. (4.2)).
- A4 Assemble into a single linear system. Each distinct collocation point determines a distinct row.
- A5 Implement boundary constraints.
- A6 Solve.

Our experience is that CM as described here is less memory intensive and executes in less time than FEM. As with FEM, it may be rendered as packaged software.

## 5. Convergence

Returning to the notation developed above, in this section we consider PDE where  $L = \lambda I - K$ . We begin with a technical lemma for linear operators between Banach spaces [1]. This result uses the standard result, if  $K$  is a bounded linear operator between Banach spaces with norm less than 1, then  $I - K$  is invertible. In addition  $(I - K)^{-1} \leq \frac{1}{1 - \|K\|}$ . For the case at hand, if  $L = \lambda I - K$  is bounded and  $\lambda > \|K\|$ , then  $L$  is invertible.

**Lemma 5.1.** *Let  $E$  be a triangle of maximal side length  $h$ . Suppose that  $Y$  is a Banach space of functions defined on  $E$ ,  $X$  is a closed subspace of  $Y$ ,  $L: X \rightarrow Y$  is a bounded, nonsingular linear operator and  $P$  is a bounded operator on  $Y$ . If  $L = \lambda I - K$  and  $\|K - PK\| \rightarrow 0$  as  $h \rightarrow 0$ , then  $\lambda I - PK$  is invertible on  $X$  and there exists a  $k > 0$  and a constant  $M$ , depending on  $k, P$  and  $K$  such that  $\|(\lambda I - PK)^{-1}\| < M$  for all  $h < k$ .*

**Proof.** By hypothesis, there is a real  $K$  such that for all  $h < K$ ,  $\|(\lambda I - K)^{-1}\| \|K - PK\| < 1$ . Hence,  $\|(\lambda I - K)^{-1}(K - PK)\| < 1$ . Therefore,

$$\begin{aligned} \lambda I - PK &= (\lambda I - K) + (K - PK) \\ &= (\lambda I - K)[I + (\lambda I - K)^{-1}(K - PK)] \end{aligned}$$

is the product of invertible linear operators and hence, invertible. Taking the norm of the inverse of both sides of this equation yields

$$\begin{aligned} \|(\lambda I - PK)^{-1}\| &\leq \|(\lambda I - K)^{-1}\| \\ &\quad \times \|[I + (\lambda I - K)^{-1}(K - PK)]^{-1}\| \\ &= M \end{aligned}$$

for sufficiently small  $k$ . In particular,  $m$  is independent of the particular choice of  $h$ .  $\square$

We now turn to the convergence theorem. In our context the spaces from the lemma are  $x = C^2(E_e)$ ,  $Y = C^0(E_e)$  and  $P = \mathcal{P}_{Z_e}^h$  with values in  $V_e^h \subset C^2(E_e)$ . Recall that the collocation solution may be expressed at the element level provided we keep in mind that the element vertices are included in the set of collocation points. In particular, (4.1) gives rise to

$$\mathcal{P}_{Z_e}^h Lu_P - \mathcal{P}_{Z_e}^h f = 0 \quad \text{for each } e. \quad (5.1)$$

**Theorem 5.2.** *Suppose the PDE is of the form  $Lv + f$ . Suppose also that for each  $E_e$ ,  $L$  on  $C^2(E_e)$  is bounded and nonsingular. If the PDE has a  $C^2$  solution  $u$  and  $\|K - \mathcal{P}_{Z_e}^h K\|_{e,\infty} \rightarrow 0$  as  $h \rightarrow 0$ , then the collocation solution  $u_P$  converges to  $u$  order  $h^2$ .*

**Proof.** We must prove that  $\|u - u_P\|_\infty < Ch^2$  for a constant  $C$  independent of the finite element partition. We will prove this for each element and then conclude the result for  $D$ . Hence, in the following argument  $u$ ,  $u_P$  and  $f$  are restricted to an element  $E_e$  and  $K$  is restricted to  $C^2(E_e)$ .

Begin with  $(\lambda I - K)u = f$ , apply  $\mathcal{P}_{Z_e}^h$  and add  $\lambda u$  to both sides get

$$\lambda u + \lambda \mathcal{P}_{Z_e}^h u - \mathcal{P}_{Z_e}^h Ku = \lambda u + \mathcal{P}_{Z_e}^h f.$$

Rearranging terms, we have

$$(\lambda I - \mathcal{P}_{Z_e}^h K)u = \lambda(u - \mathcal{P}_{Z_e}^h u) + \mathcal{P}_{Z_e}^h f.$$

Next, we subtract the expression for the collocation solution as given in (5.1),  $\lambda \mathcal{P}_{Z_e}^h u_P - \mathcal{P}_{Z_e}^h Ku_P = \mathcal{P}_{Z_e}^h f$ . The result is to cancel the terms with  $f$ .

$$\begin{aligned} (\lambda I - \mathcal{P}_{Z_e}^h K)u - \lambda u_P + \mathcal{P}_{Z_e}^h Ku_P \\ = \lambda(u - \mathcal{P}_{Z_e}^h u), \end{aligned}$$

since  $\mathcal{P}_{Z_e}^h u_P = u_P$ . We render this expression in a form that compares  $u$  to the collocation solution on the left with the polynomial interpolation in the right. In particular,

$$(\lambda I - \mathcal{P}_{Z_e}^h K)(u - u_P) = \lambda(u - \mathcal{P}_{Z_e}^h u)$$

or

$$u - u_P = \lambda(\lambda I - \mathcal{P}_{Z_e}^h K)^{-1}(u - \mathcal{P}_{Z_e}^h u),$$

by the lemma. By taking norms of both sides and using the bound from the lemma, we have the desired expression.  $\square$

The preceding convergence theorem is restated from [1] for spatial dimension greater than 1. The theorem plays a role similar to Cea's theorem for FEM. As is usual for convergence theorems, it depends on the existence of a solution  $u$ . The result is restricted as it applies only to the case  $L = \lambda I - K$ . For this theorem it is necessary to assume that  $L$  is bounded and nonsingular. Analogously, in the case of FEM, the elliptical hypothesis implies that the weak form of the PDE is bounded in each argument. Further, for  $|\lambda|$  sufficiently large,  $L$  is nonsingular. Additionally, for uniformly bounded elliptical equations with  $|\lambda|$  sufficiently large, the elliptical hypothesis is satisfied [3]. Hence, these two conditions can be thought of as analogous to the usual hypotheses for FEM convergence. The final condition is that  $K - PK$  converges to zero in norm. This condition is special to this result. In [1] the authors prove that this will hold provided  $K$  is a compact operator.

Next we consider the relationship between the CM solution and the FEM solution. We begin by considering the collocation solution as an element of the Hilbert space of square integrable functions on  $D$  with norm  $\|v\|_2 = [\int_D v^2]^{1/2}$ .

**Corollary 5.3.** *With the hypotheses of Theorem 5.2, the CM solution converges to  $u$  order  $h^2$  in  $L^2$  norm.*

**Proof.** From Theorem 5.2, we know that there is a constant  $K$  with  $\|u - u_P\|_\infty < Kh^2\|u\|_\infty$ . Since the integral is summable across the finite element partition, we have from Theorem 5.2,

$$\begin{aligned} \|u - u_P\|_2^2 &= \int_D (u - u_P)^2 \\ &= \sum_e \int_{E_e} (u - u_P)^2 \\ &\leq K^2 h^4 (\|u\|_\infty)^2 \sum_e A(E_e), \end{aligned}$$

where  $A(E_e)$  denotes the element area. Setting the area of  $D = A = \sum_e A(E_e)$ , we have

$$\|u - u_P\|_2 \leq \sqrt{AK} h^2 \|u\|_\infty. \quad \square$$

For FEM, the degree 5 model for triangles is  $C^1(D)$  and twice weakly differentiable. Hence, the function space for this model is the Sobolev space  $H_V^2$ , where  $V$  is the interior of  $D$ . If the PDE is elliptical and satis-

fies the elliptical hypothesis and if the refinement process is quasi-uniform (as described above), then the FEM solution also converges to  $u$  order  $h^2$  in  $L^2$  norm. Hence, the CM as developed here produces quantitatively equivalent results to FEM. On the other hand, for the degree 2 model, FEM convergence is only order  $h$ . We expect that when Theorem 5.2 is extended to more general equations, the order of convergence will be less.

## 6. Examples I: Comparing discontinuous collocation to FEM

We look at three PDE: the Laplace equation, the Helmholtz equation and a diffusion/transport equation. In each case we select a model and execute the problem both in FEM and CM. The setting for the Laplace equation is heat diffusion. For the Helmholtz equation we model a standing wave in a rectangular shaped pool. In the third case we look at the mean first exit time for a geometric Brownian motion. In each case we considered only a static setting so as to isolate the treatment of the spatial process. Further, for each case we test both the degree two model (6 DoF) and the degree 5 model (21 DoF).

The executions were on an HP Pavilion dv5 Notebook. The processor was an Intel i5 executing at 2.67 GHz. The operating system was Windows 7. The programming was done in *Mathematica* version 7. This is an interpreted 4GL. For the FEM integration step we use variable substitution against a reference element.

In the following we report our findings. First we make some general comments:

- (1) *CM is faster than FEM.* For the degree 5 model, FEM ran well over an hour. During the execution the processor was frequently maximally utilized. The hardware configuration was unable to process the third example. For the degree 2 model, FEM required run times of 10–15 min. In all cases CM ran to completion within a minute.
- (2) *CM executes finer partitions.* Because of the variable substitution logic in the integration step, FEM failed to process triangulations when a triangle of small area was present. The problem does not occur for CM. In order to provide comparable examples, the we had to select partitions without small triangles.

- (3) *CM is easier to program.* The programs for CM are shorter with shorter development time.
- (4) *Supporting convergence theory for CM is less complete than the corresponding theory for FEM.* Only the second example is supported by the provided theory. The first example is supported by [4]. The third example is supported by [2] for rectangular elements and successfully implemented by [6] for that partition.
- (5) *Degree 5 CM is ill-posed.* We were unable to successfully implement CM for the degree 5 model. In all cases the resulting linear system was singular.
- (6) *The results are comparable.* Even two slightly different implementations of the same technique produce distinct outputs. Hence, there is little value in comparing results directly. Nevertheless, the output for CM and FEM are qualitatively comparable.
- (7) For the examples we have used a random number generator to select the interior collocation points. Even when making no attempt to choose the best collocation points, the discontinuous CM performs similarly to FEM.

(A) Laplace equation (2-D heat),  $\nabla^2 u = 0$ .

For this example we used a notched rectangular region. The boundary values were set to 200 at the notch and zero elsewhere.

The partition (Fig. 2).

FEM graphical output (Figs 3 and 4).

CM graphical output (Fig. 5).

(B) Poisson equation (Helmholtz static wave),  $\nabla^2 u + \lambda u = 0$ .

The partition (Fig. 6).

FEM graphical output (Figs 7 and 8).

CM graphical output (Fig. 9).

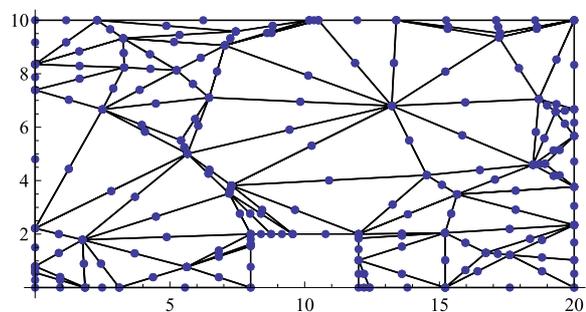


Fig. 2. Finite element partition. (Colors are visible in the online version of the article; <http://dx.doi.org/10.3233/RDA-2012-0076>.)

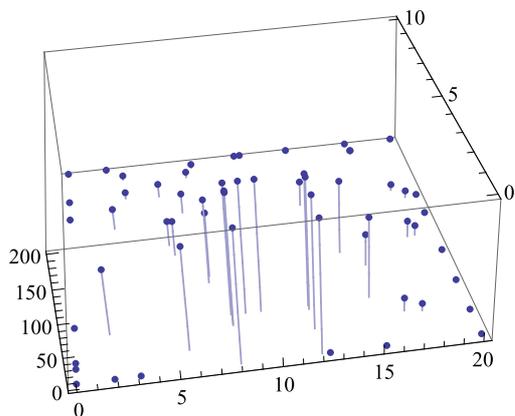


Fig. 3. FEM degree 5. (Colors are visible in the online version of the article; <http://dx.doi.org/10.3233/RDA-2012-0076>.)

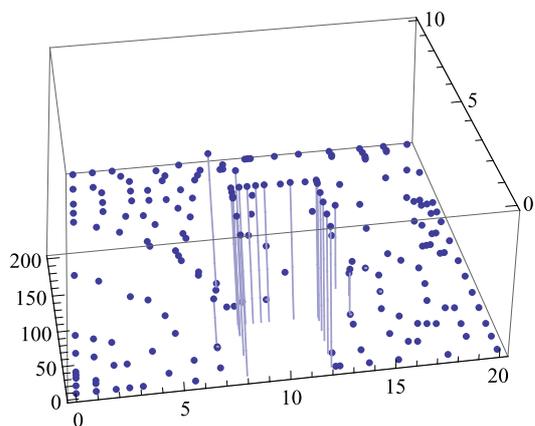


Fig. 4. FEM degree 2. (Colors are visible in the online version of the article; <http://dx.doi.org/10.3233/RDA-2012-0076>.)

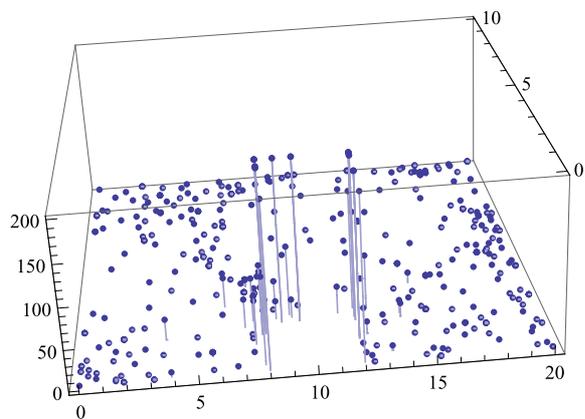


Fig. 5. CM degree 2. (Colors are visible in the online version of the article; <http://dx.doi.org/10.3233/RDA-2012-0076>.)

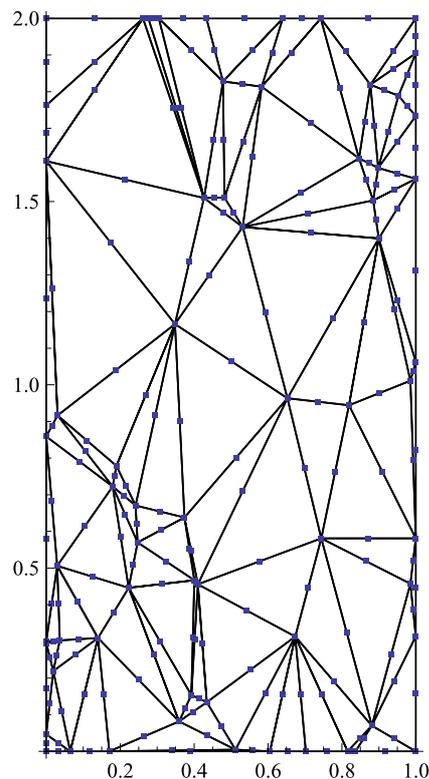


Fig. 6. Finite element partition. (Colors are visible in the online version of the article; <http://dx.doi.org/10.3233/RDA-2012-0076>.)

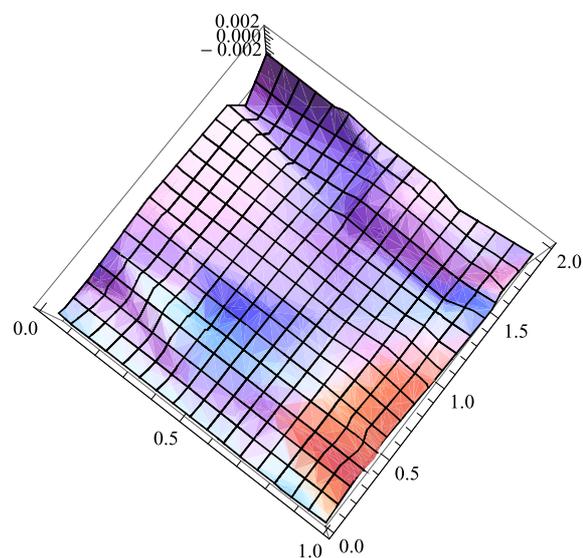


Fig. 7. FEM degree 5. (Colors are visible in the online version of the article; <http://dx.doi.org/10.3233/RDA-2012-0076>.)

(C) Transport diffusion,  $\frac{1}{2}(\sigma_1^2 x^2 \frac{\partial^2 u}{\partial x^2} + \sigma_2^2 y^2 \frac{\partial^2 u}{\partial y^2}) + b_1 x \frac{\partial u}{\partial x} + b_2 y \frac{\partial u}{\partial y} + 1 = 0$ , for diffusion constants  $\sigma_i$  and

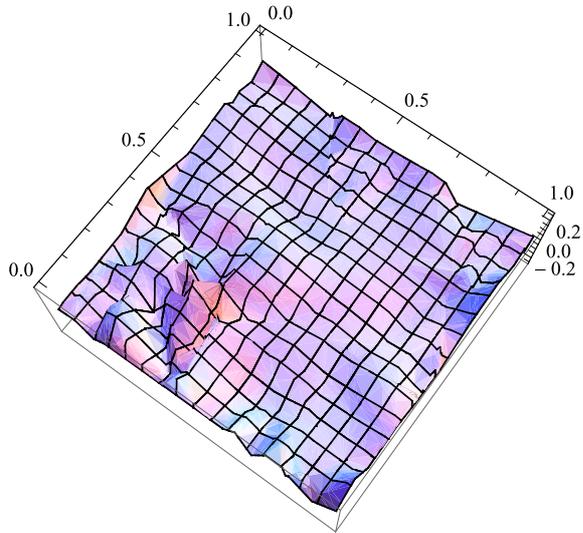


Fig. 8. FEM degree 2. (Colors are visible in the online version of the article; <http://dx.doi.org/10.3233/RDA-2012-0076>.)

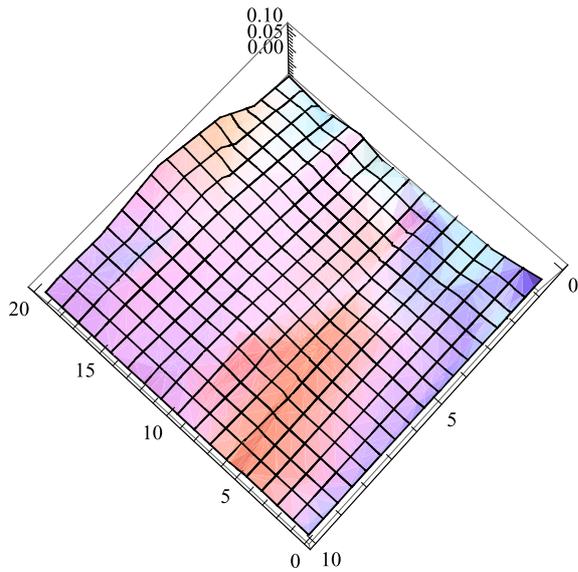


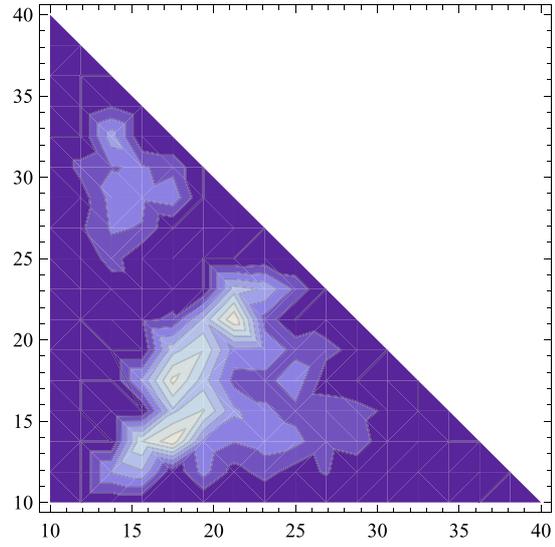
Fig. 9. CM degree 2. (Colors are visible in the online version of the article; <http://dx.doi.org/10.3233/RDA-2012-0076>.)

drift constants  $b_i$

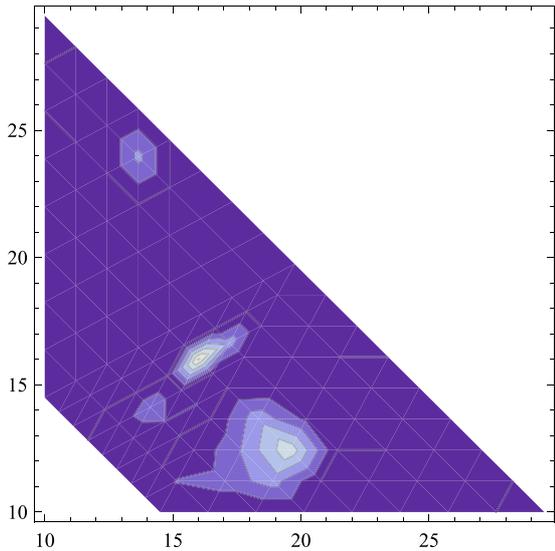
All executions based on a triangular a finite element partition were ill-posed.

### 7. Examples II: Financial examples

(A) Mean exit time from a Price Corridor. We begin this section with the equation that arose in example C



(a)



(b)

Fig. 10. (a) Triangular region, contour plot. (b) Trapezoidal region, contour plot. (Colors are visible in the online version of the article; <http://dx.doi.org/10.3233/RDA-2012-0076>.)

in Section 6,

$$\frac{1}{2} \left( \sigma_1^2 x^2 \frac{\partial^2 u}{\partial x^2} + \sigma_2^2 y^2 \frac{\partial^2 u}{\partial y^2} \right) + b_1 x \frac{\partial u}{\partial x} + b_2 y \frac{\partial u}{\partial y} + 1 = 0.$$

Consider a basket of two underlyings, for instance common stocks denoted  $x$  and  $y$ . The model requires

Table 1  
Exit times,  $y = 16$

$y = 16$	Triangle	Trapezoid
$x = 11$	0.460515	0.177355
$x = 12$	0.460922	0.203097
$x = 13$	0.0988595	0.302182
$x = 14$	0.690253	0.266983
$x = 15$	0.633036	0.374373
$x = 16$	0.178103	0.31145
$x = 17$	0.164911	0.248337
$x = 18$	1.08997	0.185414
$x = 19$	1.19645	0.207052
$x = 20$	1.60255	1.4522
$x = 21$	1.67874	1.02541
$x = 22$	1.92209	1.24737
$x = 23$	1.8356	1.36398
$x = 24$	0.606032	N/A
$x = 25$	0.894751	N/A

that they have independent price distributions. In practice this requirement is frequently relaxed. We set the dispersion constants to the volatility  $\sigma_i$ , and the drift constants to  $b_i = (D_i - r)$  where  $D_i$  represents the expected dividend and  $r$  is the interest rate returned by a safe investment. If  $D_i - r > 0$ , then the return on investment is positive and hence, the tendency will be to hold shares even if the market value decreases slightly. If we solve the equation over a region then  $u(x, y)$  will estimate the expected time to the first exit of the basket from the price region. If the region represents the knock-out boundaries of a barrier option, then  $u$  will estimate that effective life of the option.

For parameters we take

$$\begin{aligned} \sigma_1 &= 0.2, & \sigma_2 &= 0.3, \\ r &= 0.1, & D_1 &= 0.02, & D_2 &= 0.03. \end{aligned}$$

We consider two regions, first the triangle bounded by the lines

$$x = 10, \quad y = 10 \quad \text{and} \quad x + y = 40,$$

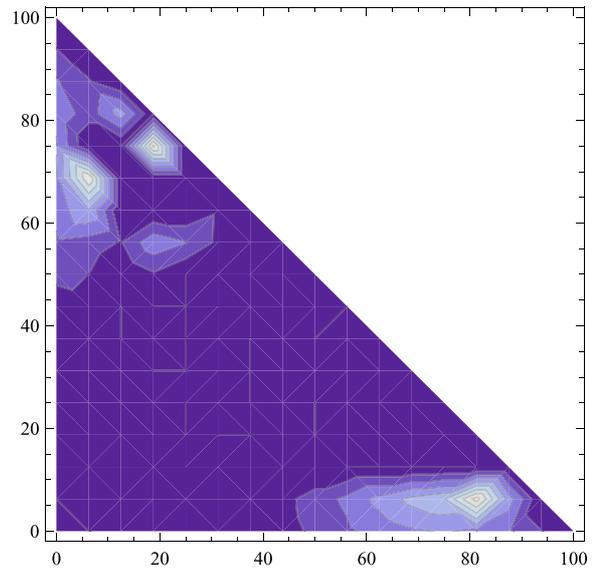
and the trapezoid

$$x = 10, \quad y = 10, \quad x + y = 15$$

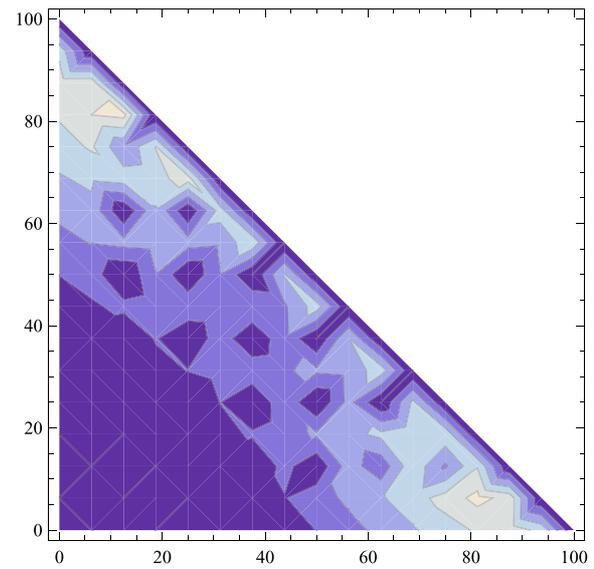
and

$$x + y = 30.$$

In each case we use a regular triangular partition with a degree 2 model. For the triangle there are 64 elements.



(a)



(b)

Fig. 11. (a) Black-Scholes, triangular region,  $T = 0.5$ . (b) Black-Scholes, triangular region,  $T = 0.05$ . (Colors are visible in the online version of the article; <http://dx.doi.org/10.3233/RDA-2012-0076>.)

The trapezoid model has 48 elements. The collocation points are the triangle vertices and the edge midpoints. The data is stated as a fraction of a year. As the model is discontinuous, we have averaged the two values at the midpoints before reporting the data.

From the contour plot we see that the maximal values occur along the diagonal  $x = y$ . Recall that a

Table 2  
Output for a triangular region

$x \setminus y$	5.1	7.6	10.1	12.6	15.1
$T = 0.5$					
5.1	NP	0.000019	0.000018	0.417605	0.430839
7.6	NP	0.000018	0.000017	0.427071	0.444369
10.1	NP	0.000169	0.000017	0.438367	0.458206
12.6	2.01444	2.03017	2.10006	5.11721	5.26693
15.1	2.081	2.10359	2.18019	8.27377	8.45718
$T = 0.05$					
5.1	NP	NP	NP	0.980758	0.988094
7.6	NP	NP	NP	0.972146	0.988909
10.1	NP	NP	NP	2.87998	0.988909
12.6	0.672178	0.6781	0.705769	2.89832	1.53642
15.1	0.698	0.706461	0.735832	0.990698	2.94073

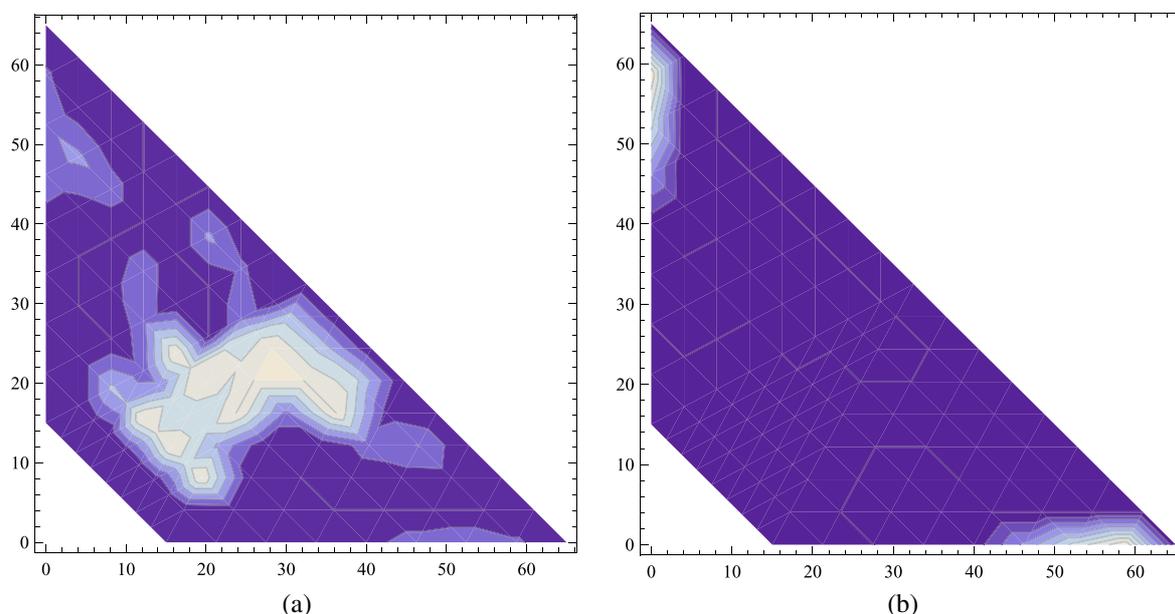


Fig. 12. (a) Black–Scholes, trapezoidal region,  $T = 0.5$ . (b) Black–Scholes, trapezoidal region,  $T = 0.05$ . (Colors are visible in the online version of the article; <http://dx.doi.org/10.3233/RDA-2012-0076>.)

two product basket with equal valued and uncorrelated items will function like a well balanced and diverse portfolio. Hence, it is expected that the value of the balanced portfolio (basket) will remain relatively stationary during a period of small market fluctuation (Fig. 10).

We report output for a line  $y = 16$  (Table 1).

(B) Barrier options. In this example we look at four cases for the Black–Scholes equation for a European call option with barrier. We include two price regions and two alternatives for the maturity,  $T = 0.5$  (a half

year) and  $T = 0.1$ . The two variable Black–Scholes equation is

$$\frac{1}{2} \left( \sigma_1^2 x^2 \frac{\partial^2 u}{\partial x^2} + \sigma_2^2 y^2 \frac{\partial^2 u}{\partial y^2} \right) + \rho \sigma_1 \sigma_2 xy \frac{\partial^2 u}{\partial x \partial y} + b_1 x \frac{\partial u}{\partial x} + b_2 y \frac{\partial u}{\partial y} - ru = -\frac{\partial u}{\partial t},$$

where  $\rho$  denotes the correlation coefficient,  $\sigma_i$  are the dispersion constants,  $b_i$  denote the drift and  $r$  is interest.

Table 3  
Output for a trapezoidal region

$x \setminus y$	16.1	18.6	21.1	23.6	26.1
$T = 0.5$					
5.1	0.246098	0.348454	0.037847	0.902848	0.935415
7.6	1.61146	0.239843	0.868252	0.888226	0.914216
10.1	1.54668	0.592032	0.397789	0.420581	0.455007
12.6	0.761302	0.647758	0.632376	0.427508	0.468151
15.1	1.30179	0.780146	0.762463	0.426859	1.04833
$T = 0.05$					
5.1	NP	NP	NP	0.119171	0.128372
7.6	NP	NP	NP	0.12514	0.13428
10.1	NP	NP	NP	0.118471	0.131464
12.6	NP	NP	NP	0.114607	0.126135
15.1	NP	NP	NP	0.116112	0.26384

As in the prior example, the estimation procedure will be discontinuous collocation with triangular elements and a degree two polynomial interpolation. For the time stepping procedure we use implicit FDM with 1000 time steps. As in the prior example the region is divided into 64 elements for the triangle case and 48 for the trapezoid. Execution times are under two minutes.

The parameter values are

$$\begin{aligned} \sigma_1 &= 0.2, & \sigma_2 &= 0.3, & \rho &= 0.5, \\ r &= 0.1, & D_1 &= 0.0, & D_2 &= 0.0, \\ \text{Strike} &= 40. \end{aligned}$$

Figure 11 shows the output for the first case. As in the prior example, the basket tends to be more volatile when one of the underlying dominates the other. Hence, the higher option price at the boundaries.

Table 2 shows representative data generated from the collocation. “NP” indicates that the price for a single share is less than  $10^{-5}$ . In these cases the probability that the basket will exceed the strike in the time to maturity is very small. The data is computed using a weighted average of the values generated at the col-

location points from the corresponding triangular elements. Further, we chose price pairs  $(x, y)$  out of the money.

We present the contour plots for the trapezoidal region (Fig. 12). For the longer maturity time, the results look most like the estimated exit time indicating that the portfolio is should be stable.

Table 3 shows data generated from the collocation for the trapezoidal region.

### References

- [1] K. Atkinson and W. Han, *Theoretical Numerical Analysis*, 2nd edn, Springer, New York, 2005.
- [2] B. Bialecki, Convergence analysis of orthogonal spline collocation for elliptical boundary value problems, *SIAM J. Numer. Anal.* **35**(2) (1998), 617–631.
- [3] L.C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, Vol. 19, Amer. Math. Soc., Providence, RI, 1998.
- [4] Z.C. Li, T.T. Lu, H.Y. Hu and H.D. Cheng, *Trefftz and Collocation Methods*, WIT Press, Southampton/Boston, 2008.
- [5] G. Phillips, *Interpolation and Approximation of Polynomials*, Springer, New York, 2003.
- [6] J. Topper, *Financial Mathematics with Finite Elements*, Wiley, Chichester, 2005.